



mathematical models and methods

unit 32

Partial differential
equations



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An Inter-faculty Second Level Course
MST204 Mathematical Models and Methods

Unit 32

Partial differential equations

Prepared for the Course Team
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Study guide

Before reading this unit you should read those sections of *Unit 25* which refer to the calculation of partial derivatives.

You should read Sections 1–3 before watching the television programme, but you can choose to read Section 4 either before or after the programme.

1 Introduction and definitions

1.1 Introduction

This unit is devoted almost entirely to what can truly be described as one of the ‘gems’ of late eighteenth and early nineteenth century mathematics—the solution of partial differential equations with given boundary conditions—although in the space available we shall be able to examine only one simple case in any detail.

My objective will be to give you a glimpse of the power of the methods which you have studied so far—and incidentally, and more prosaically, to enable you to revise some of the topics of this course in what I hope is an interesting fashion.

The subject which we shall study arose in a period when a number of great mathematicians were turning their minds to the investigation of various questions arising in what we should call *applied mathematics*. To their credit, the men of this age would have drawn no clear distinction between what was ‘pure’ and what ‘applied’; they were simply intent on obtaining answers to interesting questions, and in searching for these answers they incidentally laid the foundation for modern *mathematical analysis* as well as for modern physics.

A catalogue of the names associated with the subject reads like an honours list: Euler, Laplace, Lagrange, Poisson, Cauchy, the Bernoullis, Fourier, and many others. Before we become involved in the details of the single example which we have time to study, it will perhaps give you a deeper understanding of the relevance of the topic if I say just a little about the range of subjects which these men considered.

A number of apparently distinct areas of physics—electrostatics, magnetism, hydrodynamics, aerodynamics, heat conduction and wave motion—use models which involve differential equations with partial derivatives. Such an equation is called a **partial differential equation**. The equation

$$\frac{\partial^2 \theta}{\partial x^2} = k \frac{\partial \theta}{\partial t}$$

is an example of a partial differential equation, and it can be used to model the flow of heat along a thin metal bar; in this case $\theta(x, t)$ is the temperature at position x at time t . Having deduced that θ satisfies such an equation, we are very little further forward unless we are able to determine θ explicitly, using any available information about the temperature at some particular point in time and how heat may be lost from the system.

The break-through in such problems came with the development of trigonometric series and the publication of Joseph Fourier’s *Theorie Analytique de la Chaleur* (1822), which is the source of many important methods in mathematical physics, in particular for problems involving partial differential equations under given **boundary conditions**. Certainly the subject had been advanced by Euler, D’Alembert, David Bernoulli and others, but it was Fourier who made the situation absolutely clear by establishing the fact that an ‘arbitrary’ function (in fact a *piecewise continuous* function) could be expressed in the form

$$\sum_{n=0}^{\infty} (A_n \cos nax + B_n \sin nax).$$

Despite the support of Euler and Bernoulli, the idea was so original when first presented in 1807 that it raised some vigorous opposition, notably, it is said, from Lagrange (the leading French mathematician of the day). The reluctance of some

See, for example, Chapter 22 of *Mathematical Thought from Ancient to Modern Times*, by Morris Kline (Oxford University Press, 1972).

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mathematicians to accept Fourier's results must certainly have been due in part to the fact that it called into question the very meaning of the fundamental concept of a function. Ultimately, the 'Fourier series' was universally adopted, and the questions which it raised spurred mathematicians to re-examine the foundations of the calculus and to restore it in its modern form as *mathematical analysis*.

The essential elements of Fourier's method are easy to describe. First derive the appropriate partial differential equation, then assume that the required function can be expressed in terms of a Fourier series, and finally use the equation and the initial conditions to determine the constants in the series. This is precisely the outline of the core of this unit; but, before we start on this programme, I should like to look at the problem that we shall attempt to solve in more detail.

Imagine a coiled spring stretched tightly between two fixed points A and B (Figure 1). Now suppose that a point in the middle of the spring is pulled to one side, towards B , and then released, while the ends are kept fixed (Figure 2). The spring will start to vibrate, with each point of the spring moving to and fro in some fashion. Our task is to determine how each point will move, and to discover where it will be at each subsequent instant of time. In our solution of this problem there will be three main factors:

- (i) the physical properties of the spring, which will give rise to an equation known as the **wave equation**;
- (ii) the physical constraints applied to the spring, namely that it is fixed at both ends, which give rise to the so-called **boundary conditions**;
- (iii) the initial behaviour of the spring, in other words, what we do to the spring in order to make it move. This initial behaviour produces the **initial conditions** for the problem.

The exact meaning of these terms will become clear as we proceed through the unit.

Although I have chosen to focus on the motion of a vibrating spring, the methods which we shall investigate are very general, and apply to the equations used for a wide variety of physical situations including heat flow, the sound produced by musical instruments and the motion of fluids.

The essence of each of these physical situations is that it can be modelled by a **partial differential equation**, and so I must explain in more detail what I mean by such an equation.

1.2 Partial differential equations

Previously in this course we have solved a differential equation such as

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0,$$

which is an example of an **ordinary differential equation**, and is characterized as such by the fact that the unknown function $y(x)$ is a function of the single variable x . We are now intending to discuss differential equations which apply to functions of two variables. Such a function, $U(x, t)$ say, can vary both because of changes in x and because of changes in t .

As an illustration, consider the equation $x \frac{\partial U}{\partial x} = t \frac{\partial U}{\partial t}$. The function $U(x, t) = e^{xt}$ is a solution of this equation, as we can easily verify. We have

$$\frac{\partial U}{\partial x} = te^{xt} \quad \text{and} \quad \frac{\partial U}{\partial t} = xe^{xt},$$

and it follows that $x \frac{\partial U}{\partial x} = t \frac{\partial U}{\partial t}$, and so

$$U(x, t) = e^{xt}$$

A  B

Figure 1. A coiled, stretched spring.

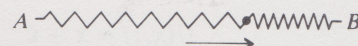
A  B

Figure 2. The same spring with the middle pulled towards B .

We shall see later how this use of the term 'boundary conditions' relates to the use in Unit 6, Section 3.

is a **solution** of the **partial differential equation**

$$x \frac{\partial U}{\partial x} = t \frac{\partial U}{\partial t}.$$

As you may have guessed, the general problem which we should like to solve is to discover *all* of the functions which satisfy a given partial differential equation, and this is the process which we call **solving the equation**. Generally speaking, solving a partial differential equation is by no means easy, and the following illustration may help you to understand why.

Suppose that we are given the rather simple looking partial differential equation

$$\frac{\partial U(x, t)}{\partial x} = 1.$$

The left-hand side is the partial derivative of $U(x, t)$ with respect to x *keeping t constant*, so to find U we may keep t constant and integrate with respect to x . We then obtain

$$U(x, t) = x + C.$$

However, the constant C is only constant if t is kept fixed, and for different values of t the value of C will be different, so that C is a function of t . Thus the most general solution is

$$U(x, t) = x + f(t)$$

where f is an *arbitrary function*. You can verify that any such function $U(x, t)$ is a solution by differentiating partially with respect to x .

So you see that while the solution of an ordinary differential equation will involve arbitrary *constants*, the solution of a partial differential equation will involve arbitrary *functions*.

As we increase the order of the partial derivatives in a partial differential equation, so we introduce more arbitrary functions into the solution. Thus the partial differential equation

$$\frac{\partial^2 U}{\partial x \partial t} = 0$$

might be expected to contain two arbitrary functions in its solution, and the following argument shows that this is indeed so.

Integrating with respect to x (for constant t) we see that $\frac{\partial U}{\partial t} = f(t)$, where f is an arbitrary function (of one variable). Now integrating this equation with respect to t (for constant x), we get

$$U(x, t) = \int f(t) dt + g(x),$$

where g is another arbitrary function. This solution can be written in the form

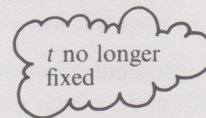
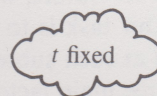
$$U(x, t) = h(t) + g(x)$$

where both h and g are arbitrary functions.

In order to make any progress it will be necessary for me to restrict the class of partial differential equations to be discussed. Partial differential equations are classified in much the same way as ordinary differential equations (see *Unit 6*), so that a **linear second-order partial differential equation** in two independent variables will be one of the form

$$a(x, t) \frac{\partial^2 U}{\partial x^2} + b(x, t) \frac{\partial^2 U}{\partial x \partial t} + c(x, t) \frac{\partial^2 U}{\partial t^2} + d(x, t) \frac{\partial U}{\partial x} + e(x, t) \frac{\partial U}{\partial t} + f(x, t) U = g(x, t)$$

where a, b, c, d, e, f and g are known functions of x and t , and at least one of a, b and c is non-zero.



This is still far too wide a class to tackle in one unit, and in fact our concern will be with equations which (in addition to obeying the above conditions) are **homogeneous** (i.e. $g(x, t)$ is the zero function) and **constant-coefficient** (i.e. a, b, c, d, e and f are all constants). Thus, our attention will be confined to equations of the form

$$a \frac{\partial^2 U}{\partial x^2} + b \frac{\partial^2 U}{\partial x \partial t} + c \frac{\partial^2 U}{\partial t^2} + d \frac{\partial U}{\partial x} + e \frac{\partial U}{\partial t} + fU = 0 \quad (1)$$

where a, b, c, d, e and f are constants.

These are the analogues of the ordinary differential equations of the form

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0$$

which are discussed in Section 1 of *Unit 6*. Theorem 1 of that section also has the following analogue for partial differential equations, which is of extreme importance to us for our work later in this unit.

Theorem 1

If u and v are two solutions of a linear homogeneous partial differential equation, then for all values of A and B , the function $Au + Bv$ is also a solution of the equation.

Proof (for the second-order constant-coefficient case)

Let $U = Au + Bv$, where u and v are solutions of Equation (1). Then

$$\begin{aligned} a \frac{\partial^2 U}{\partial x^2} &= a \frac{\partial^2}{\partial x^2} (Au + Bv) \\ &= aA \frac{\partial^2 u}{\partial x^2} + aB \frac{\partial^2 v}{\partial x^2} \\ &= A \left(a \frac{\partial^2 u}{\partial x^2} \right) + B \left(a \frac{\partial^2 v}{\partial x^2} \right) \end{aligned}$$

and similarly for the other terms in the equation. Thus

$$\begin{aligned} a \frac{\partial^2 U}{\partial x^2} + b \frac{\partial^2 U}{\partial x \partial t} + c \frac{\partial^2 U}{\partial t^2} + d \frac{\partial U}{\partial x} + e \frac{\partial U}{\partial t} + fU \\ = A \left(a \frac{\partial^2 u}{\partial x^2} + \cdots + fu \right) + B \left(a \frac{\partial^2 v}{\partial x^2} + \cdots + fv \right) \\ = 0 \quad (\text{since } u \text{ and } v \text{ are solutions of Equation (1)}), \end{aligned}$$

so that U is a solution of Equation (1).

In fact this theorem holds (as stated) for all linear homogeneous partial differential equations; the general proof is longer than the above but very similar in its method.

Before we go any further, I must make sure that you are confident with partial derivatives, so if you have any doubts about your ability try the following exercises.

Exercise 1

Find $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ for each of the following functions $F(x, y)$.

(i) $F(x, y) = \frac{x}{y}$

(ii) $F(x, y) = Ax^2 + 2Bxy + Cy^2$

(iii) $F(x, y) = \sin(2x + y)$

(iv) $F(x, y) = \phi(2x + y)$, where ϕ is an arbitrary differentiable function of one variable. (Use the notation ϕ' for the derived function of ϕ .)

(v) $F(x, y) = \phi(ax^2 + by^2)$ where ϕ is an arbitrary differentiable function of one variable.

Exercise 2

Write down simple expressions for the following:

- (i) $\lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x}$
- (ii) $\lim_{\delta x \rightarrow 0} \frac{F(x + \delta x, t) - F(x, t)}{\delta x}$
- (iii) $\lim_{\delta x \rightarrow 0} \frac{\frac{\partial F}{\partial x}(x + \delta x, t) - \frac{\partial F}{\partial x}(x, t)}{\delta x}$.

Exercise 3

Find $\frac{\partial^2 F}{\partial x^2}$, $\frac{\partial^2 F}{\partial y \partial x}$, $\frac{\partial^2 F}{\partial x \partial y}$ and $\frac{\partial^2 F}{\partial y^2}$ for each of the following functions $F(x, y)$.

- (i) $F(x, y) = xy$
- (ii) $F(x, y) = x^3 + 2x^2y + y^3$
- (iii) $F(x, y) = e^{x+y}$

Exercise 4

Verify that each of the following functions is a solution of the corresponding partial differential equation. (f and g are arbitrary differentiable functions of one variable, and a and b are given constants.)

- (i) $U = f(x)g(y); \quad U \frac{\partial^2 U}{\partial x \partial y} = \frac{\partial U}{\partial x} \frac{\partial U}{\partial y}$
- (ii) $U = f(ay - bx); \quad a \frac{\partial U}{\partial x} + b \frac{\partial U}{\partial y} = 0$
- (iii) $y = f(x - t) + g(x + t); \quad \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$.

Exercise 5

Determine which of the following partial differential equations are homogeneous linear, and which are constant-coefficient.

- (i) $x^2 \frac{\partial^2 F}{\partial x^2} + y^2 \frac{\partial^2 F}{\partial y^2} = 0$
- (ii) $xy \frac{\partial^2 F}{\partial x^2} + x \frac{\partial F}{\partial x} \frac{\partial F}{\partial y} + y \frac{\partial^2 F}{\partial y^2} = 0$
- (iii) $x^2 \left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} \right) - \frac{\partial F}{\partial x} - \frac{\partial F}{\partial y} = e^{xy}$
- (iv) $\frac{\partial^2 F}{\partial x^2} = a^2 \frac{\partial F}{\partial t}$ (where a is a real constant)

[Solution to Exercises 1–5 on p. 35]

Summary of Section 1

- In this section you have seen examples of **partial differential equations** and their **solutions**, and you have seen that such solutions involve *arbitrary functions*, as opposed to the arbitrary constants which occur in the solution of ordinary differential equations.
- In this unit we shall restrict our attention to **homogeneous, constant-coefficient, linear, second-order** partial differential equations; i.e. equations of the form

$$a \frac{\partial^2 U}{\partial x^2} + b \frac{\partial^2 U}{\partial x \partial t} + c \frac{\partial^2 U}{\partial t^2} + d \frac{\partial U}{\partial x} + e \frac{\partial U}{\partial t} + fU = 0$$

where a, b, c, d, e and f are constants.

- Theorem 1:** If u and v are two solutions of any homogeneous, linear partial differential equation, then for any constants A and B , the function $Au + Bv$ is also a solution of the equation.

2 Properties of coiled springs and elastic strings

2.1 Hooke's law

A coiled spring is a device, usually made of steel, which can be altered in length, but which returns to its original shape when any external forces are removed. Such a spring may be extended, in which case we say that it is in a **state of tension**, or possibly compressed, which we call a **state of compression**. Our objective in this section is to model the behaviour of such springs in some detail, and in order to do this we shall need to make certain simplifying assumptions.

The springs which you have met previously in this course were assumed to be *ideal springs*, that is to say, they were assumed to obey Hooke's law and to be *light*. (By 'light' we mean that the mass of the spring can be ignored because it is small relative to the other quantities under discussion.) You will recall that **Hooke's law** states that an ideal spring when uniformly extended will exert a force F , given by

$$F = K(l - l_0)$$

where K is the **stiffness** of the spring, l_0 is its natural length and l is its extended length. We shall find it convenient to replace K by $\frac{k}{l_0}$, and the constant k is then known as the **modulus of elasticity** of the spring, or more briefly its **modulus**. We then have

$$F = \frac{k}{l_0}(l - l_0).$$

The springs under consideration in this section are no longer assumed to be light, but I shall make the assumption that they continue to obey Hooke's law. (This is an assumption which can be validated only by experiment.) In future in this unit all references to 'springs' are intended to mean *heavy springs which obey Hooke's law*.

Later in this section I propose to model the behaviour of a spring which is originally stretched horizontally between two fixed points. The spring is then disturbed slightly from its rest position, and our task will be to predict its subsequent motion. The essential point to note at this stage is that our starting point could well be a spring which has already been extended, but before I continue with this line of thought I must define the term **tension in the spring**.

We consider a stretched spring with ends A and B which may, or may not, be stationary, and consider an arbitrary point P along its length (see Figure 1). The section of spring to the right of P exerts a force towards B , of magnitude T say, on the left-hand section; and from Newton's third law we deduce that the left-hand section applies an equal and opposite force on the right-hand section. The magnitude of this force is called the **tension in the spring at P** . The value of T will in general depend upon the choice of the point P , but we shall assume throughout our discussion that the spring is always stretched rather than compressed, so that T is always positive.

My first task is to examine the relationship between the tension in the spring and its physical shape.

The springs which I shall consider will be long compared to their width, so that in future diagrams I shall represent them by straight lines.

Suppose that the spring is now stretched between two fixed points A and B a distance L apart, and that the spring is stationary. This situation will in future be known as the **reference state** of the spring.

While in its reference state the tension at all points of the spring is the same, T_0 say, for each small element of the spring must experience equal and opposite forces at its ends, and therefore the tension must be the same at both ends of the element. (If this were not so, the element would begin to move in the direction of the greater force.)

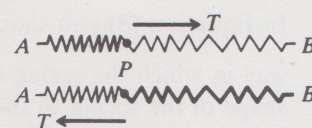


Figure 1. The forces acting at P on the left-hand and right-hand sections of a stretched spring.

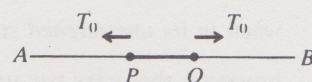


Figure 2. The forces at P and Q must be equal.

We now choose an arbitrary point P on the spring (and mark it say with a white dot), and our objective will be to describe how P moves when the spring is disturbed. Suppose initially that while the spring is in its reference state the point P is at a distance x from A . Suppose also that Q is a point close to P , at a distance $x + \delta x$ from A while the spring is in its reference state.

The element of the spring between P and Q of length δx had an original unstretched length of δa , say, so that applying Hooke's law to this small section of spring we obtain

$$T_0 = k \left(\frac{\delta x - \delta a}{\delta a} \right).$$

We can solve this equation for δa and thus find an expression for the unstretched length of the element PQ in the form

$$\delta a = \left(\frac{k}{T_0 + k} \right) \delta x. \quad (1)$$

Let us now consider what happens if the spring is set in motion by a small disturbance along its length. Suppose that at time t the point P on the spring has moved so that its distance from A is s , and further suppose that Q is a distance $s + \delta s$ from A . Using Hooke's law again we have

$$T \simeq k \left(\frac{\delta s - \delta a}{\delta a} \right), \quad (2)$$

since the length of the element PQ is now δs , and T is the tension at P . (This equation is approximate since we assume that the spring is uniformly extended between P and Q in order to apply Hooke's law.)

Now, eliminating δa between Equations (1) and (2), we obtain

$$T \simeq (T_0 + k) \frac{\delta s}{\delta x} - k,$$

so that in the limit as $\delta x \rightarrow 0$ we eliminate the local effects of non-uniform stretching of the spring and obtain an exact result

$$T = (T_0 + k) \frac{ds}{dx} - k. \quad (3)$$

(For the time being I am regarding t as fixed, so that it is unnecessary to use a partial derivative in Equation (3).)

In Equation (3) you should notice that $\frac{ds}{dx}$ is a quantity which depends upon the way in which the spring is distorted, so that this equation relates the geometric shape of the spring to the tension at any point of the spring. Also I have not actually used the condition that A and B are fixed, so that Equation (3) would also apply in the case where A and B are free to move. The above equation is used in the following example, which is intended to get you accustomed to the idea of tension varying along the length of a spring.

Example 1

A stationary uniform spring of mass M_0 , modulus k and unstretched length l_0 hangs vertically and supports a weight of mass M . Find the length of the stretched spring.

Solution

We are told that the spring is *uniform*, which indicates that its mass per unit length when in its unstretched state is constant and equal to $\frac{M_0}{l_0}$. (When stretched,

however, the mass per unit length is no longer constant because the stretching is not performed evenly along its length.) The solution now falls into two parts: an application of Newton's second law, then an application of Hooke's law.

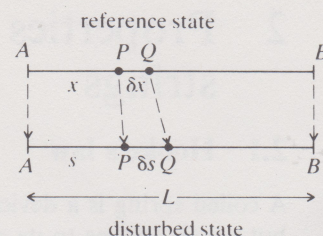


Figure 3. The reference state and a disturbed state of a stretched spring.

(i) Apply Newton's second law to a small element of the spring

We choose the variables as shown in Figure 4, and consider the forces acting on a small element δs of the stretched spring.

The left-hand side of the figure shows the spring in its reference state (with gravity 'switched off') which corresponds to the case in which $T_0 = 0$ and the spring is unstretched. This element was originally of length δx , and its mass is therefore

$$\frac{M_0}{l_0} \delta x.$$

Suppose now that the tension in the stretched spring is T_P at the point P and T_Q at the point Q . The element PQ experiences three forces (see Figure 5):

- (i) its weight downwards,
- (ii) the tension at P upwards,
- (iii) the tension at Q downwards,

giving a net downwards force of

$$T_Q - T_P + \frac{M_0 g}{l_0} \delta x.$$

Since we are assuming that the stretched spring is stationary, the acceleration of the element PQ is zero, and from Newton's second law we have

$$\text{force} = \text{mass} \times \text{acceleration},$$

so that

$$T_Q - T_P + \frac{M_0 g}{l_0} \delta x = 0.$$

We now write the difference in tension between Q and P as δT , i.e.

$$\delta T = T_Q - T_P.$$

It follows that

$$\frac{\delta T}{\delta x} = -\frac{M_0 g}{l_0},$$

and in the limit,

$$\frac{dT}{dx} = -\frac{M_0 g}{l_0}.$$

Now integrating with respect to x ,

$$T(x) = -\frac{M_0 g x}{l_0} + C, \quad (5)$$

where C is a constant. $T(x)$ is thus the tension at the point P which was at a distance x from A when the spring was in its reference state.

In order to determine C we consider the forces acting on the mass M :

- (i) a force upwards of $T_B = T(l_0) = -\frac{M_0 g l_0}{l_0} + C = C - M_0 g$,
- (ii) a force downwards of Mg ;

so that the net force downwards is

$$Mg + M_0 g - C,$$

and this is equal to zero since the mass M is stationary. It follows that

$$C = (M + M_0)g.$$

(You should notice in passing that we could also have found the value of C by examining the tension at the point A and equating this to the total weight hanging from A .)

Here, unlike our previous situation, the 'unstretched state' is our 'reference state'.

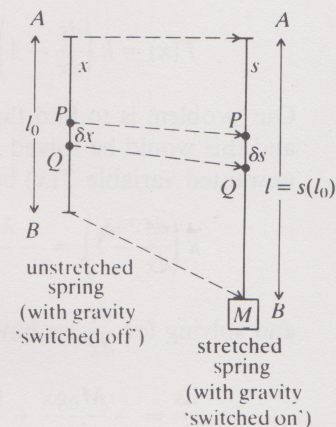
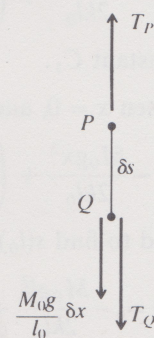


Figure 4. The spring in its unstretched and stretched states.



(4) Figure 5. The forces acting on PQ .

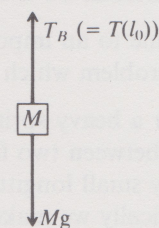


Figure 6. The forces acting on the mass M .

Equation (5) now becomes

$$T(x) = -\frac{M_0 g x}{l_0} + (M + M_0)g. \quad (6)$$

Thus Newton's second law tells us how the tension $T(x)$ depends on x .

(ii) *Apply Hooke's law*

From Equation (3) on page 10 we have, since $T_0 = 0$ in the reference state we have chosen,

$$T(x) = k \left(\frac{ds}{dx} - 1 \right). \quad (7)$$

Our problem is to find the length of the stretched spring, which is $s(l_0) = l$ say, and this would be solved if we could find the function $s(x)$. Eliminating the unwanted variable $T(x)$ between (6) and (7), we obtain

$$k \left(\frac{ds}{dx} - 1 \right) = -\frac{M_0 g x}{l_0} + (M + M_0)g,$$

and solving for $\frac{ds}{dx}$ we have

$$\frac{ds}{dx} = -\frac{M_0 g x}{kl_0} + \frac{(M + M_0)g}{k} + 1.$$

Integrating with respect to x , we obtain

$$s = -\frac{M_0 g x^2}{2kl_0} + \left(\frac{(M + M_0)g}{k} + 1 \right) x + C_1$$

for some constant C_1 .

But $s = 0$ when $x = 0$, and therefore $C_1 = 0$, so that

$$s = -\frac{M_0 g x^2}{2kl_0} + \left(\frac{(M + M_0)g}{k} + 1 \right) x. \quad (8)$$

We are asked to find $s(l_0)$, so substituting $x = l_0$ into (8) we have

$$\begin{aligned} s(l_0) &= -\frac{M_0 g l_0^2}{2kl_0} + \left(\frac{(M + M_0)g}{k} + 1 \right) l_0 \\ &= \left(\frac{(M + \frac{1}{2}M_0)g}{k} + 1 \right) l_0. \end{aligned}$$

When you derive an answer to a problem like this it is often useful to check its validity for extreme cases. Thus, for example, we see that if k is very large, corresponding to the case of a very stiff spring, then from Equation (8) we obtain $s \simeq x$, which is just what we should expect.

2.2 Vibrations of a stretched spring

We now come to an important point in this unit, namely the point at which we define the problem which we shall eventually solve.

We consider a heavy spring, at first in its reference state, stretched horizontally and tightly between two fixed points A and B at a distance L apart, and then disturbed by small longitudinal forces which impart a small motion to the spring. More specifically we make the following assumptions.

- (i) The motion of any section of the spring is small compared to the spring's length, takes place entirely along the length of the spring, and the end points are fixed.
- (ii) All frictional forces, such as that due to air resistance, may be neglected.
- (iii) The forces due to the tension in the spring act along its length and obey Hooke's law.

- (iv) The gravitational force on the spring is small relative to the tension, and so may be neglected.
- (v) The diameter of the spring is small compared to its length, and so may be neglected.

Suppose that we choose an arbitrary point P of the spring, whose reference position is at a distance x from A , and suppose that the spring is disturbed slightly and begins to move. Our objective is to determine the position of the point P at any subsequent time. More specifically, suppose that the point P has moved to a position at a distance s from A at time t , then $s = s(x, t)$ is some function of x and t , and our task will be to determine this function. Suppose also that the point Q close to P , whose reference position is at a distance $x + \delta x$ from A , has moved to a position at a distance $s + \delta s$ from A at time t (Figure 7).

Before the spring began to move it was in its reference state stretched uniformly between A and B , and thus its mass per unit length was a constant, m say. (This is not equal to the *unstretched* mass per unit length.)

In order to model the motion of the spring we need to bring together two ingredients, as in Example 1:

- (i) Newton's second law (which specifies the acceleration if the force is known),
- (ii) Hooke's law (which determines the tension).

We let T_P be the tension at P and T_Q be the tension at Q , at time t .

(i) *Apply Newton's second law to a small element of the spring*

The small element of the spring between P and Q was originally of length δx , so that its mass is $m\delta x$. This element is acted on by two opposing forces, the tension at P and the tension at Q . The centre of mass of this element is somewhere between P and Q and, since the element is small, its acceleration is approximately

$$\frac{\partial^2 s}{\partial t^2}(x, t) \quad \text{to the right.}$$

(This corresponds to the acceleration of a particular point of the spring, i.e. a fixed value of x , and hence we keep x fixed in the differentiation.) From Newton's second law we have

$$\text{force} = \text{mass} \times \text{acceleration},$$

so that

$$T_Q - T_P \simeq m\delta x \frac{\partial^2 s}{\partial t^2},$$

and therefore

$$\frac{\delta T}{\delta x} \simeq m \frac{\partial^2 s}{\partial t^2}.$$

In the limit as $\delta x \rightarrow 0$ the centre of mass of the element becomes P , so that

$$\frac{\partial T}{\partial x} = m \frac{\partial^2 s}{\partial t^2}. \quad (9)$$

(ii) *Apply Hooke's law*

We now use Equation (3) on page 10 (derived from Hooke's law):

$$T = (T_0 + k) \frac{ds}{dx} - k.$$

The derivative in this equation refers to a fixed value of t . Now that we are

treating t as a variable we must write $\frac{\partial s}{\partial x}$ to indicate that t is held constant during the differentiation, and we then have

$$T = (T_0 + k) \frac{\partial s}{\partial x} - k. \quad (10)$$

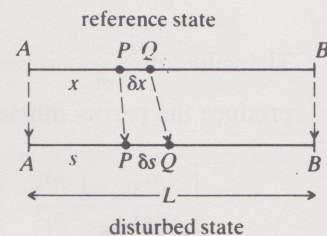


Figure 7. The spring in its reference state and a disturbed state.

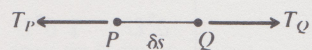


Figure 8. The forces acting on a small element of the spring.

Differentiating (10) partially with respect to x , we obtain

$$\frac{\partial T}{\partial x} = (T_0 + k) \frac{\partial^2 s}{\partial x^2}. \quad (11)$$

We wish to discover how s depends on x and t , and therefore we want to eliminate the superfluous variable T . We can do this by equating Equations (9) and (11) to obtain

$$m \frac{\partial^2 s}{\partial t^2} = (T_0 + k) \frac{\partial^2 s}{\partial x^2}. \quad (12)$$

The value of $\frac{T_0 + k}{m}$ is positive, and it is common practice to put $c^2 = \frac{T_0 + k}{m}$ to produce the partial differential equation

$$\frac{\partial^2 s}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 s}{\partial t^2}. \quad (13)$$

This equation is known as the (one-dimensional) **wave equation**, and is of crucial importance for the rest of this unit. In fact it is this partial differential equation that you will learn how to solve. The constant c is called the **wave speed**.

The function $s = s(x, t)$ determines the distance of a point P from A at any time t ; but it is more common to write the equation in terms of $U = s - x$, so that $U = U(x, t)$ determines the distance of P from its starting point (see Figure 9).

If we substitute $s = U + x$ into Equation (13), we obtain

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2}. \quad (14)$$

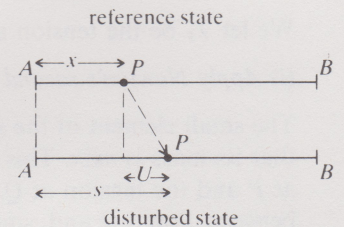


Figure 9. Derivation of the function $U(x, t)$.

Exercise 1

Verify that the substitution of $s = U + x$ into Equation (13) gives Equation (14).

[Solution on p. 35]

The advantage of using U rather than s is that the conditions which follow from the assumption that the ends of the spring are fixed are quite simply stated as

$$\left. \begin{aligned} U(0, t) &= 0 \\ U(L, t) &= 0 \end{aligned} \right\} \text{ for all } t.$$

These conditions are called the **boundary conditions** for the partial differential equation, and will be used extensively in Section 3.

2.3 Elastic strings

An **elastic string** is just what it says it is, a string with elastic properties; but as a mathematical term it specifies that the string satisfies Hooke's law. Springs and strings are therefore identical in behaviour except in one important respect; a string cannot be in a state of compression.

We introduce the concept of an elastic string simply to give you another illustration of the use of the wave equation, and one which is perhaps a little easier to visualize. We suppose that the string is initially stretched tightly between two fixed points A and B , and then disturbed slightly in a direction perpendicular to the length of the string (see Figure 10).

If P is a point on the string at a distance x to the right of A , then it can be shown that $U(x, t)$, the perpendicular displacement of P from the line AB , will satisfy the wave equation

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2}$$

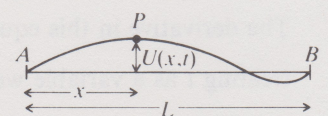


Figure 10. An elastic string vibrating.

where c is some constant, dependent on the physical properties of the string and its initial tension. (I ask you to accept this on trust since I feel that it would take too much of your time to give yet another derivation of the wave equation.)

In this case the boundary conditions are again

$$\left. \begin{aligned} U(0, t) &= 0 \\ U(L, t) &= 0 \end{aligned} \right\} \text{for all } t.$$

As in the case of a spring, the task is to determine the function $U(x, t)$ from the initial behaviour of the string.

In order to distinguish the two forms of motion, vibrations along the length of the string or spring are called **longitudinal vibrations** and vibrations across a string at right angles to its length are called **transverse vibrations**.

Summary of Section 2

1. In this section we modelled the behaviour of a 'heavy' spring, and we used Hooke's law to derive the equation

$$T = (T_0 + k) \frac{ds}{dx} - k.$$

2. By combining this result with Newton's second law we showed that with certain assumptions the longitudinal displacement U of a stretched spring satisfied the **wave equation**

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2}$$

for some constant c , where U satisfies the **boundary conditions**

$$\left. \begin{aligned} U(0, t) &= 0 \\ U(L, t) &= 0 \end{aligned} \right\} \text{for all } t.$$

3. We introduced the terms **longitudinal** and **transverse vibrations**, and noted that transverse vibrations of a string also satisfy the wave equation and the same boundary conditions.

3 The method of separation of variables

You will recall that I have set myself the task of finding a function which determines the position of a vibrating spring at any time t . In other words, I have to find a specific solution of the wave equation

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2}$$

satisfying the boundary conditions

$$U(0, t) = U(L, t) = 0 \quad \text{for all } t.$$

It is clear that this equation as it stands does not contain sufficient information, for at no stage in the previous section have I introduced the initial position and velocity of the spring. However, as we shall see, if the initial position and velocity are specified we can find a definite solution to the problem.

Before we get deeply involved in the details of the methods which we use to solve partial differential equations, I should like to say a little about the general strategy that we adopt. The point is that in general it is really rather difficult to find *all* the solutions of a partial differential equation. If, however, we restrict our search for solutions to those of a particularly simple form, then we may be able to reduce the problem from a difficult *partial differential equation* to a *number of* much easier *ordinary differential equations*. Such a process is the method which is known as **separation of variables**, and I shall explain the details of it shortly.

Let us suppose for a moment that we do have a method which will enable us to find *some* of the solutions of the wave equation

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2},$$

say $U_1, U_2, \dots, U_n$; then since this partial differential equation is homogeneous linear, we know that any linear combination

$$\sum_{i=1}^n A_i U_i$$

(where $A_1, A_2, \dots, A_n$ are constants) is also a solution, from Theorem 1 on page 7.

The method of separation of variables does indeed provide us with such a set of solutions, and by taking linear combinations we can obtain many more. But this is an idea which we shall develop later; at the moment our concern is to find *some* solutions of the wave equation.

3.1 Separation of variables

Our immediate plan is to find *some* solutions of the wave equation

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2} \quad (1)$$

satisfying the boundary conditions

$$\left. \begin{aligned} U(0, t) &= 0 \\ U(L, t) &= 0 \end{aligned} \right\} \text{for all } t.$$

We now take 'a leap in the dark', guided only by the fact that other mathematicians, wiser than ourselves, have seen the final outcome, and attempt to find some solutions of Equation (1) by assuming that they take the form

$$U(x, t) = X(x)T(t). \quad (2)$$

In other words, U is the product of two functions X and T , where X is a function of x alone and T is a function of t alone. (There is of course at this stage no guarantee whatever that there are any solutions which take this form. By constructing them we shall in fact show that they exist.)

Do not confuse this use of T , to indicate a function of time, with our previous use of this letter to denote the tension in a spring.

Differentiating Equation (2) partially with respect to x , we obtain

$$\frac{\partial U(x, t)}{\partial x} = X'(x)T(t)$$

where X' denotes the derived function of X . (Remember that T , being a function of t only, can be regarded as a constant for this calculation.) Differentiating partially with respect to x again, and abbreviating the notation,

$$\frac{\partial^2 U}{\partial x^2} = X''T. \quad (3)$$

Similarly, differentiating the original Equation (2) twice with respect to t , we obtain

$$\frac{\partial^2 U}{\partial t^2} = XT''. \quad (4)$$

Substituting Equations (3) and (4) into (1), we get

$$X''T = \frac{1}{c^2} XT''.$$

Now dividing both sides by XT to get all the functions of t on one side and functions of x on the other, we obtain

$$\frac{T''}{c^2 T} = \frac{X''}{X}$$

(assuming that neither X nor T is zero). This then is a necessary condition that $U = XT$ should be a solution of the original equation.

The next step caused me considerable difficulty as a student, and I needed to think about it for quite a while. We let

$$\mu = \frac{T''}{c^2 T} = \frac{X''}{X};$$

then since $\mu = \frac{T''}{c^2 T}$ (a function of t only), it must be independent of x ; but since

$\mu = \frac{X''}{X}$ (a function of x only), it must be independent of t . But if μ is independent of x and t , it can only be a constant.

[I hope that I have made the point more clearly than it was made to me, but, if

you still find it difficult, let $\mu(x, t) = \frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)}$. Then

$$\mu(x, t) = \frac{T''(t)}{c^2 T(t)} = \mu(0, t) \quad (\text{since each expression is independent of } x)$$

and

$$\mu(0, t) = \frac{X''(x)}{X(x)} = \mu(0, 0) \quad (\text{since each expression is independent of } t).$$

Hence

$$\mu(x, t) = \mu(0, 0) \text{ for all } x \text{ and } t,$$

and therefore μ is constant.]

It follows that any solution of the original partial differential equation which is of the required form must satisfy the two ordinary differential equations

$$T'' = c^2 \mu T \quad \text{and} \quad X'' = \mu X. \quad (5)$$

We know that μ is a real (rather than a complex) constant (because T and T'' are real), and therefore there are three cases to consider:

$$\mu > 0, \quad \mu = 0 \quad \text{and} \quad \mu < 0.$$

Before I discuss the three cases in general, try the following exercises, the first to prepare you for the general argument and the second because the result will be useful later.

Exercise 1

Solve the equations

$$T'' = c^2 \mu T \text{ and } X'' = \mu X$$

in each of the following cases:

(i) $c = 2, \quad \mu = 1$

(ii) $c = 2, \quad \mu = -1$

(iii) $c = -2, \mu = 1.$

Exercise 2

If $h \neq 0$, and $Ae^{ht} + Be^{-ht} = 0$ for all values of t , show that $A = B = 0$.

[Solutions to Exercises 1 and 2 on p. 36]

Remember that we are trying to solve the partial differential equation

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2}$$

with $U(0, t) = U(L, t) = 0$, and we have seen that a solution of the form

$U(x, t) = X(x)T(t)$ must satisfy $T'' = c^2 \mu T$ and $X'' = \mu X$. Now we use the

boundary conditions to filter out the solutions which are of no interest in the three cases $\mu > 0$, $\mu = 0$ and $\mu < 0$.

The process is rather like panning for gold — we need to filter out quite a lot of useless material, until finally we are left with a nugget at the bottom of the bowl.

Case I: $\mu > 0$

For convenience we can put $\mu = h^2$ where h is a real constant; then the equations

$$T'' = c^2 h^2 T \quad \text{and} \quad X'' = h^2 X$$

have solutions

$$T = Ae^{cht} + Be^{-cht} \quad \text{and} \quad X = Ce^{hx} + De^{-hx}$$

where A, B, C and D are constants.

Now if *either* $T(t) = 0$ for all t or $X(x) = 0$ for all x , then $U(x, t) = 0$ for all x, t , and although this is in fact a solution of the partial differential equation, it is a rather boring solution since it corresponds to the spring remaining stationary. The solution $U(x, t) = 0$ is called the **trivial solution**. We are actually looking for *non-trivial* solutions which satisfy the boundary conditions

$$U(0, t) = U(L, t) = 0 \quad \text{for all } t.$$

These boundary conditions refer to particular values of x (namely 0 and L) and all values of t , and assert that

$$\left. \begin{aligned} (Ce^0 + De^0) T(t) &= 0 \\ (Ce^{hL} + De^{-hL}) T(t) &= 0 \end{aligned} \right\} \text{for all } t.$$

As we are searching for non-trivial solutions, we may assume that $T(t) \neq 0$ for *some* value of t , and this allows us to conclude that

$$\begin{aligned} C + D &= 0 \\ Ce^{hL} + De^{-hL} &= 0. \end{aligned}$$

The unique solution to this pair of equations is $C = D = 0$, which leads to the trivial solution after all, in spite of all our efforts to avoid it!

Case II: $\mu = 0$

In this case the equations for T and X become

$$T'' = 0 \quad \text{and} \quad X'' = 0.$$

As in Case I, all that really matters about T is our assumption that $T(t) \neq 0$ for some t . The solution of $X'' = 0$ is

$$X = Cx + D,$$

and if we again apply the boundary conditions we obtain

$$\left. \begin{aligned} DT(t) &= 0 \\ (CL + D)T(t) &= 0 \end{aligned} \right\} \text{for all } t.$$

Our assumption that $T(t) \neq 0$ for some t implies that

$$\begin{aligned} D &= 0 \\ CL + D &= 0 \end{aligned}$$

so that $C = D = 0$, and again the only permissible solution is the trivial solution.

Case III: $\mu < 0$

For convenience we put $\mu = -h^2$; then

$$X'' = -h^2 X \quad \text{and} \quad T'' = -c^2 h^2 T. \quad (6)$$

As in part (ii) of Exercise 1 we see that the solution of the first of these equations is

$$X = C \cos hx + D \sin hx.$$

We now apply the boundary conditions (again simply assuming that T is not the zero function) to obtain

$$\left. \begin{aligned} CT(t) &= 0 \\ (C \cos hL + D \sin hL) T(t) &= 0 \end{aligned} \right\} \text{for all } t,$$

and then choose a value of t for which $T(t) \neq 0$. We deduce that $C = 0$ and therefore

$$D \sin hL = 0.$$

We may assume that $D \neq 0$ (for otherwise we are once again left with the trivial solution), and therefore

$$\sin hL = 0.$$

This is the 'gold nugget' for which we have been searching. If $\sin hL = 0$, then it follows that

$$hL = n\pi, \quad \text{i.e. } h = \frac{n\pi}{L},$$

for some integer n .

Solving the second equation of (6), we obtain

$$T = A \cos cht + B \sin cht,$$

which gives us the solution

$$\begin{aligned} U(x, t) &= X(x)T(t) \\ &= D \sin hx(A \cos cht + B \sin cht) \end{aligned}$$

where $h = \frac{n\pi}{L}$ for some integer n .

Substituting this expression for h and absorbing the arbitrary constant D into the arbitrary constants A and B , we finally have a solution

$$U(x, t) = \sin \frac{n\pi x}{L} \left(A \cos \frac{cn\pi t}{L} + B \sin \frac{cn\pi t}{L} \right)$$

which satisfies the boundary conditions and which is certainly not the trivial solution. More than that, we have discovered an infinite number of distinct solutions corresponding to the choices of the integer n :

$$U_1(x, t) = \sin \frac{\pi x}{L} \left(A_1 \cos \frac{c\pi t}{L} + B_1 \sin \frac{c\pi t}{L} \right)$$

$$U_2(x, t) = \sin \frac{2\pi x}{L} \left(A_2 \cos \frac{2c\pi t}{L} + B_2 \sin \frac{2c\pi t}{L} \right)$$

etc., and the general function of this type would be

$$U_n(x, t) = \sin \frac{n\pi x}{L} \left(A_n \cos \frac{nc\pi t}{L} + B_n \sin \frac{nc\pi t}{L} \right)$$

where n is an integer, and the A s and B s are arbitrary constants.

These solutions resemble the normal modes of a lumped-parameter vibrating system, and what we shall do next corresponds to taking linear combinations of normal mode solutions to get the general solution. Although we have indeed found rather a lot of solutions of the wave equation, we have by no means found them all, and it will need some further work to increase our store of solutions.

Unit 24

Exercise 3

Use the method of separation of variables to show that if

$$\theta(x, t) = X(x)T(t)$$

is a solution of the partial differential equation

$$\frac{\partial^2 \theta}{\partial x^2} = c^2 \frac{\partial \theta}{\partial t},$$

then

$$X'' = c^2 \mu X \quad \text{and} \quad T' = \mu T$$

for some constant μ .

(This partial differential equation may be used to model the flow of heat along a thin metal rod, in which case $\theta(x, t)$ represents the temperature at time t at a distance x from one end of the rod.)

[Solution on p. 36]

3.2 Initial conditions

In the last subsection we discovered an infinite number of solutions of the wave equation (each satisfying the appropriate boundary conditions):

$$U_n(x, t) = \sin \frac{n\pi x}{L} \left(A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L} \right)$$

where $n = 1, 2, 3, \dots$

The current state of our investigation is summarized in the set diagram, where:

$\mathcal{S} = \{\text{all solutions of the wave equation}\},$

$\mathcal{B} = \{\text{all solutions satisfying the boundary conditions } U(0, t) = U(L, t) = 0\},$

$\mathcal{P} = \{\text{all solutions of the product form } U(x, t) = X(x)T(t)\}.$

We have discovered that $\mathcal{P} \cap \mathcal{B}$, the intersection of the sets $\mathcal{P}$ and $\mathcal{B}$, consists of solutions $\{U_n(x, t)\}$; but our objective was to discover all the elements of the set $\mathcal{B}$ (shaded in Figure 1).

We now use Theorem 1 (page 7), which tells us that we may add two solutions of the wave equation and obtain a third solution, but there is still the question of the boundary conditions. In fact if we add two solutions which satisfy the conditions

$$U(0, t) = U(L, t) = 0 \text{ for all } t,$$

then we obtain a third solution with this property. For this reason these particular boundary conditions are called **homogeneous**. (We shall see an example of non-homogeneous boundary conditions in Section 4.)

In fact any function of the form

$$U(x, t) = \sum_{n=1}^N U_n(x, t),$$

for some integer N , is an element of the set $\mathcal{B}$.

We are now ready to introduce the final factor into our investigation:

The initial conditions

In order to specify the motion of the spring, we need to know where it is and how it is moving at the start of the motion, say at time $t = 0$. To know where it is we need the function $U(x, 0)$, which tells us the displacement of each point of the

spring when $t = 0$. To know how it is moving we need the function $\frac{\partial U}{\partial t}(x, 0)$,

which tells us the velocity of each point of the spring when $t = 0$.

If we are given $U(x, 0)$ and $\frac{\partial U}{\partial t}(x, 0)$, then this should be sufficient information for us to pick out the function $U(x, t)$ from all the other functions in the set $\mathcal{B}$, and thus to determine the motion of the spring. Let us now see how this works in practice.

Example 1

A uniform heavy spring stretched horizontally between the points $x = 0$ and $x = \pi$ is given an initial displacement

$$U(x, 0) = \sin x$$

along the length of the spring and released from rest. Find its subsequent displacement as a function of x and t , if the wave speed (i.e. the constant appearing in the wave equation) is c .

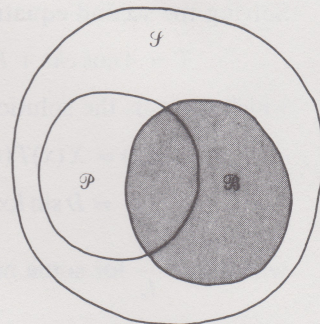


Figure 1. Various sets of solutions of the wave equation.

The connection between the two senses of the word 'homogeneous' (as applied to partial differential equations and as applied to boundary conditions) is that they both imply the property that 'the sum of two solutions is another solution'.

Solution

We can apply the wave equation

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2}$$

for some constant c , and we try a solution of the form

$$\begin{aligned} U(x, t) &= \sum_{n=1}^N U_n(x, t) \\ &= \sum_{n=1}^N \sin nx (A_n \cos nct + B_n \sin nct) \end{aligned} \quad (7)$$

for some integer N , where A_n and B_n are constants which have yet to be determined.

Note the simplification that results in this example from choosing the length L to be equal to π .

We are told that

$$U(x, 0) = \sin x, \quad (8)$$

and since the spring starts from rest we have

$$\frac{\partial U}{\partial t}(x, 0) = 0. \quad (9)$$

Putting $t = 0$ in (7), and using (8), we obtain

$$\sin x = A_1 \sin x + A_2 \sin 2x + A_3 \sin 3x + \cdots + A_N \sin Nx. \quad (10)$$

Then differentiating (7) partially with respect to t , putting $t = 0$ and then using (9), we obtain

$$0 = c(B_1 \sin x + 2B_2 \sin 2x + 3B_3 \sin 3x + \cdots + NB_N \sin Nx). \quad (11)$$

Our task is now to choose the A s and B s in order to satisfy Equations (10) and (11), and I have chosen this example deliberately so that it is possible to see these values at once. All we have to do is to choose

$$A_1 = 1, A_n = 0 \text{ for } n > 1, \text{ to satisfy Equation (10),}$$

and

$$B_n = 0 \text{ for all } n, \text{ to satisfy Equation (11).}$$

Equation (7) now reduces to

$$U(x, t) = \sin x \cos ct$$

and this is the solution to our problem.

At different instants of time t , the function $U(x, t)$ tells us the displacement of the point on the spring whose reference position is at a distance x from A .

If we choose a particular point P on the spring, so that x is regarded as constant, then P moves backward and forward under the influence of the factor $\cos ct$. The amplitude of the vibration is determined by the factor $\sin x$, so that the movement is, as we should expect, most violent at the centre of the spring.

Figure 2 (overleaf) shows the positions of various points of the springs at various times; for example $P_1, P_2, \dots, P_7$ represent various positions of the moving point P at times $t_1, t_2, \dots, t_7$.

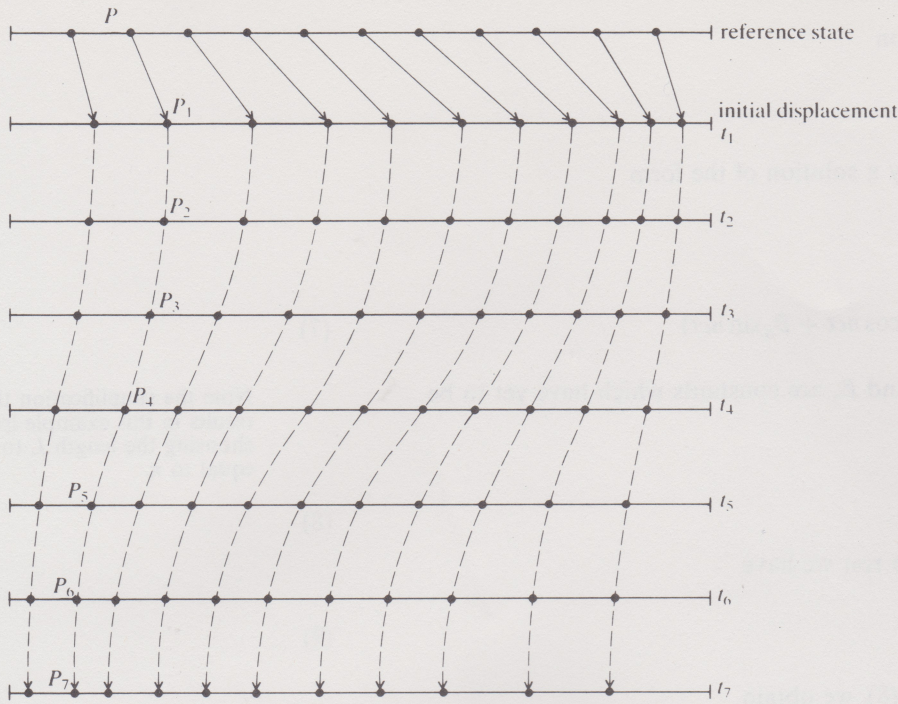


Figure 2. The movement of a longitudinally vibrating spring.

It can be seen from this example that our task is finally reduced to finding the coefficients in two expressions similar to Fourier sine series, and thus at last we see the link between this unit and the preceding one. Using the techniques developed in *Unit 31*, we should be able to solve our problem in general.

3.3 Finding the coefficients in the Fourier series

Suppose that ϕ and ψ are given functions, each defined between 0 and L , with the properties that

$$\phi(0) = \phi(L) = \psi(0) = \psi(L) = 0,$$

and we wish to find a solution of the partial differential equation

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2}$$

on the interval $0 \leq x \leq L$, which satisfies the boundary conditions

$$U(0, t) = U(L, t) = 0 \quad \text{for all } t$$

and the initial conditions

$$U(x, 0) = \phi(x) \quad \text{if } 0 \leq x \leq L, \quad (12)$$

$$\frac{\partial U}{\partial t}(x, 0) = \psi(x) \quad \text{if } 0 \leq x \leq L. \quad (13)$$

Using the results of Subsections 3.1 and 3.2, we can attempt to find a solution in the form of an infinite series

$$\begin{aligned} U(x, t) &= \sum_{n=1}^{\infty} U_n(x, t) \\ &= \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left(A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L} \right). \end{aligned} \quad (14)$$

From Equation (12), and putting $t = 0$ in the above series, we obtain

$$\phi(x) = U(x, 0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}. \quad (15)$$

From Equation (13), and differentiating the series in Equation (14) partially with respect to t , we have

$$\frac{\partial U}{\partial t}(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left(-\frac{n\pi c}{L} A_n \sin \frac{n\pi c t}{L} + \frac{n\pi c}{L} B_n \cos \frac{n\pi c t}{L} \right),$$

and therefore

$$\psi(x) = \frac{\partial U}{\partial t}(x, 0) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n \sin \frac{n\pi x}{L}. \quad (16)$$

We now use the techniques of Unit 31, Section 4, to determine the constants A_n and B_n , and to do this we use the functions ϕ and ψ (which are non-periodic) to construct two new functions, Φ and Ψ , which (like the sine function) are odd functions and periodic. We let

$$\begin{aligned} \Phi(x) &= \phi(x) & \text{if } 0 \leq x \leq L \\ &= -\phi(-x) & \text{if } -L \leq x \leq 0 \end{aligned}$$

and

$$\begin{aligned} \Psi(x) &= \psi(x) & \text{if } 0 \leq x \leq L \\ &= -\psi(-x) & \text{if } -L \leq x \leq 0 \end{aligned}$$

and further require that both functions Φ and Ψ are periodic with period $2L$ (see Figure 3).

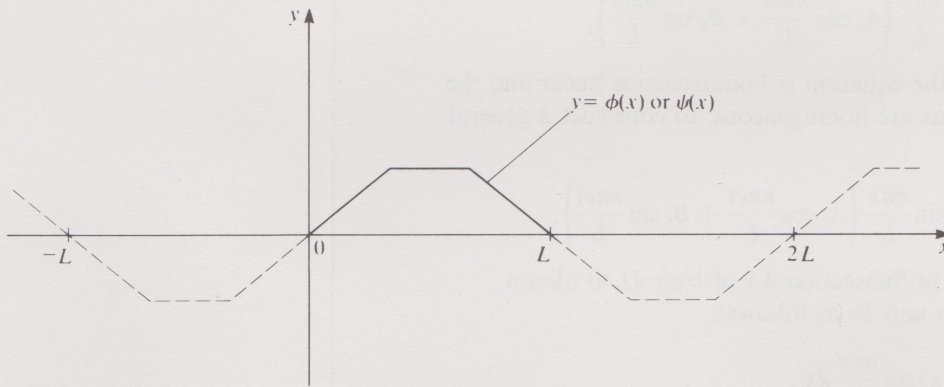


Figure 3. Obtaining an odd function of period $2L$ from a function defined between 0 and L .

We now use the formula for A_n given in Subsection 4.1 of Unit 31, allowing for the fact that the period of our function is $2L$, not L . We obtain from Equation (15):

$$\begin{aligned} A_n &= \frac{1}{L} \int_{-L}^L \Phi(x) \sin \frac{n\pi x}{L} dx \\ &= \frac{1}{L} \left(\int_0^L \Phi(x) \sin \frac{n\pi x}{L} dx - \int_0^{-L} \Phi(x) \sin \frac{n\pi x}{L} dx \right) \\ &= \frac{1}{L} \left(\int_0^L \phi(x) \sin \frac{n\pi x}{L} dx + \int_0^{-L} \phi(-x) \sin \frac{n\pi x}{L} dx \right) \\ &= \frac{1}{L} \left(\int_0^L \phi(x) \sin \frac{n\pi x}{L} dx + \int_0^L \phi(r) \sin \frac{n\pi r}{L} dr \right) \end{aligned}$$

(putting $r = -x$ in the second integral).

Thus

$$A_n = \frac{2}{L} \int_0^L \phi(x) \sin \frac{n\pi x}{L} dx.$$

Similarly we obtain from Equation (16):

$$\frac{n\pi c}{L} B_n = \frac{2}{L} \int_0^L \psi(x) \sin \frac{n\pi x}{L} dx,$$

i.e.

$$B_n = \frac{2}{n\pi c} \int_0^L \psi(x) \sin \frac{n\pi x}{L} dx.$$

To summarize, the general procedure for solving the wave equation with homogeneous boundary conditions is given in the box below.

Procedure 3.3

Given the wave equation

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2} \quad (0 \leq x \leq L)$$

subject to the boundary conditions

$$U(0, t) = U(L, t) = 0 \quad \text{for all } t$$

and the initial conditions

$$U(x, 0) = \phi(x) \quad \text{if } 0 \leq x \leq L,$$

$$\frac{\partial U}{\partial t}(x, 0) = \psi(x) \quad \text{if } 0 \leq x \leq L.$$

1. Use the method of separation of variables to obtain solutions

$$U_n(x, t) = \sin \frac{n\pi x}{L} \left(A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L} \right).$$

2. Use the facts that the equation is homogeneous linear and the boundary conditions are homogeneous, to construct a general solution

$$U(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left(A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L} \right).$$

3. Use the procedure in Subsection 4.1 of *Unit 31* to obtain formulas for the A s and B s as follows:

$$A_n = \frac{2}{L} \int_0^L \phi(x) \sin \frac{n\pi x}{L} dx$$

$$B_n = \frac{2}{n\pi c} \int_0^L \psi(x) \sin \frac{n\pi x}{L} dx.$$

If the functions ϕ and ψ are sufficiently simple, we may attempt to find formulas for A_n and B_n in terms of n ; otherwise we may compute say the first twenty coefficients, perhaps using Simpson's rule and a modern computer to calculate the integrals. In either event our problem is essentially solved.

Let us try this technique on a particular problem.

Example 2

Solve the partial differential equation

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2}$$

on the interval $0 \leq x \leq \pi$ subject to the boundary conditions

$$U(0, t) = U(\pi, t) = 0 \quad \text{for all } t,$$

and the initial conditions

$$U(x, 0) = x(\pi - x) \quad \text{if } 0 \leq x \leq \pi,$$

$$\frac{\partial U}{\partial t}(x, 0) = 0 \quad \text{if } 0 \leq x \leq \pi.$$

Solution

We attempt to find a series solution

$$U(x, t) = \sum_{n=1}^{\infty} \sin nx (A_n \cos nct + B_n \sin nct).$$

Applying the formulas in the above box for A_n and B_n , we obtain

$$A_n = \frac{2}{\pi} \int_0^{\pi} x(\pi - x) \sin nx \, dx.$$

Using the method of integration by parts, we can show that

$$\int_0^{\pi} x \sin nx \, dx = -\frac{\pi}{n} \cos n\pi$$

and

$$\int_0^{\pi} x^2 \sin nx \, dx = -\frac{\pi^2}{n} \cos n\pi + \frac{2}{n^3} (\cos n\pi - 1).$$

Thus

$$\begin{aligned} A_n &= \frac{2}{\pi} \left\{ -\frac{\pi^2}{n} \cos n\pi + \frac{\pi^2}{n} \cos n\pi - \frac{2}{n^3} (\cos n\pi - 1) \right\} \\ &= \frac{4}{\pi n^3} (1 - \cos n\pi) \\ &= \frac{4}{\pi n^3} (1 - (-1)^n). \end{aligned}$$

Since the initial conditions give $\psi(x)$ as the zero function, we have

$$B_n = 0 \quad \text{for all } n,$$

and thus we have found a solution to the problem in the form of an infinite series:

$$U(x, t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^3} (1 - (-1)^n) \sin nx \cos nct.$$

This expression can be simplified, by noticing that $1 - (-1)^n$ is equal to 0 for even n and to 2 for odd n , so that we have to sum only over odd n . Thus, putting $n = 2m - 1$, we finally obtain

$$U(x, t) = \frac{8}{\pi} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^3} \sin (2m-1)x \cos (2m-1)ct.$$

Exercise 4

Solve the partial differential equation

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2}$$

with the boundary conditions

$$U(0, t) = U(\pi, t) = 0 \quad \text{for all } t$$

and initial conditions

$$U(x, 0) = \phi(x) \quad \text{if } 0 \leq x \leq \pi$$

$$\frac{\partial U}{\partial t}(x, 0) = 0 \quad \text{if } 0 \leq x \leq \pi$$

where (with d a given positive constant)

$$\begin{aligned} \phi(x) &= \frac{2dx}{\pi} \quad \text{if } 0 \leq x \leq \frac{\pi}{2} \\ &= 2d \left(1 - \frac{x}{\pi} \right) \quad \text{if } \frac{\pi}{2} \leq x \leq \pi. \end{aligned}$$

Hint: the solution to Problem 3(i) in Unit 31, Section 5 shows that for any positive l ,

$$\frac{2}{l} \int_0^l \phi(x) \sin \frac{n\pi x}{l} dx = \frac{8d}{n^2 \pi^2} \sin \frac{n\pi}{2},$$

Just as in Example 1, the assumption that $L = \pi$ simplifies the formula.

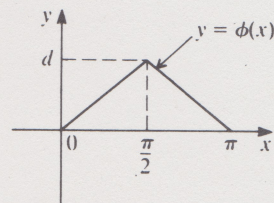


Figure 4. The graph of $\phi(x)$.

where

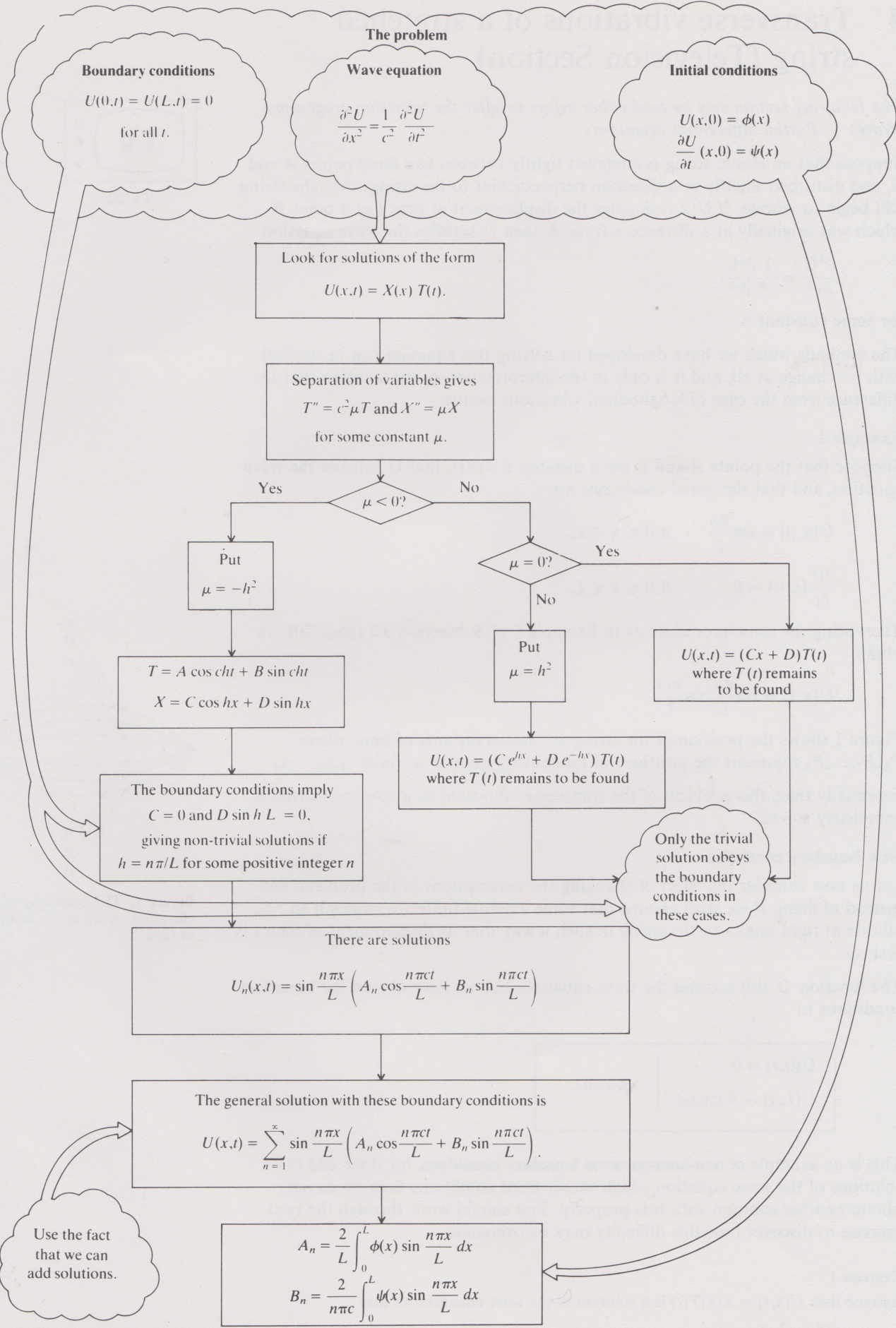
$$\phi(x) = \begin{cases} \frac{2dx}{l} & \text{if } 0 \leq x \leq \frac{l}{2} \\ 2d\left(1 - \frac{x}{l}\right) & \text{if } \frac{l}{2} \leq x \leq l. \end{cases}$$

[Solution on p. 36]

Summary of Section 3

In this section we have used the method of **separation of variables** to obtain a solution of the **wave equation** subject to certain **boundary conditions** and **initial conditions**.

The method is summarized in the following flow chart.



4 Transverse vibrations of a stretched string (Television Section)

The following section may be read either before or after the television programme 'Waves — Partial differential equations'.

Suppose that an elastic string is stretched tightly between two fixed points A and B , and disturbed slightly in a direction perpendicular to the string; then the string will begin to vibrate. If $U(x, t)$ denotes the displacement at time t of a point P which was originally at a distance x from A , then U satisfies the wave equation

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2}$$

for some constant c .

The methods which we have developed for solving this equation can be applied with no change at all, and it is only in the interpretation of the solution that the difference from the case of longitudinal vibrations occurs.

Example 1

Suppose that the points A and B are a distance L apart, that U satisfies the wave equation, and that the initial conditions are

$$U(x, 0) = \sin \frac{\pi x}{L} \quad \text{if } 0 \leq x \leq L,$$

$$\frac{\partial U}{\partial t}(x, 0) = 0 \quad \text{if } 0 \leq x \leq L.$$

Then using the same procedure as in Example 1 of Subsection 3.2 (page 20), we obtain

$$U(x, t) = \sin \frac{\pi x}{L} \cos \frac{\pi ct}{L}.$$

Figure 1 shows the position of the string at various instants of time, where $P_1, P_2, \dots, P_6$ represent the positions of the moving point P at times $t_1, t_2, \dots, t_6$.

Essentially then, this problem of the transverse vibrations of a stretched string is completely solved.

New boundary conditions

Let us now consider the effect of changing the assumptions in the problem, and instead of fixing B we shall assume that some external influence causes it to vibrate at right angles to the string in such a way that its displacement at time t is $R \sin \omega t$.

The function U still satisfies the wave equation, but we have altered the boundary conditions to

$$\left. \begin{array}{l} U(0, t) = 0 \\ U(L, t) = R \sin \omega t \end{array} \right\} \text{ for all } t.$$

This is an example of **non-homogeneous boundary conditions**, for if we add two solutions of the wave equation which satisfy these conditions then we do not obtain another solution with this property. You should work through the next exercise to discover how this difficulty may be overcome.

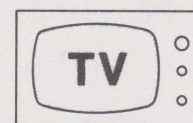
Exercise 1

Assume that $U(x, t) = X(x)T(t)$ is a solution of the wave equation, so that

$$T'' = c^2 \mu T$$

$$X'' = \mu X$$

for some constant μ .



TV 32

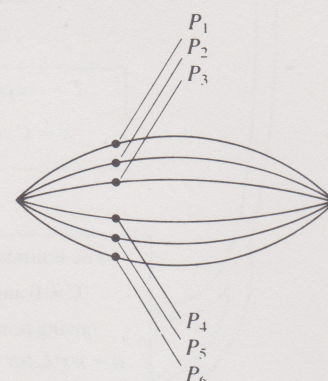


Figure 1. The positions of a vibrating string at six instants of time.

- (i) Use the boundary condition

$$U(L, t) = R \sin \omega t$$

to show that

$$T(t) = \frac{R \sin \omega t}{X(L)}.$$

- (ii) Substitute the above expression for $T(t)$ into the equation $T'' = c^2 \mu T$ and show that

$$\mu = -\frac{\omega^2}{c^2}.$$

- (iii) Solve the equation

$$X'' = \mu X$$

$$\text{with } \mu = -\frac{\omega^2}{c^2}.$$

- (iv) Use the condition $U(0, t) = 0$ to simplify the solution obtained in part (iii).

- (v) Write down a solution of the wave equation which satisfies the above (non-homogeneous) boundary conditions.

[Solution on p. 36]

At this stage you may well ask how we might find the many other solutions of the wave equation which also satisfy these boundary conditions. In fact we use a rather nice trick, and suppose that $U_0(x, t)$ is the solution which we have found in Exercise 1, and $U(x, t)$ is some other solution satisfying these boundary conditions. Then consider the function

$$V(x, t) = U(x, t) - U_0(x, t).$$

The essential point to notice is that U_0 and U satisfy the same condition at $x = L$, namely

$$U(L, t) = U_0(L, t) = R \sin \omega t \quad \text{for all } t,$$

and therefore

$$V(L, t) = U(L, t) - U_0(L, t) = 0 \quad \text{for all } t.$$

Thus $V(x, t)$ is just a solution of precisely the problem which we solved in Section 3, i.e. it satisfies the wave equation and the boundary conditions

$$V(0, t) = V(L, t) = 0 \quad \text{for all } t.$$

Thus we can find $V(x, t)$ as a series and then write

$$U(x, t) = U_0(x, t) + V(x, t)$$

to solve our present problem.

An example should make the method clearer, and for simplicity I shall choose $L = \pi$.

Example 2

Solve the wave equation

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2}$$

with boundary conditions

$$U(0, t) = 0 \quad \text{for all } t$$

$$U(\pi, t) = R \sin \omega t \quad \text{for all } t$$

subject to the initial conditions

$$U(x, 0) = 0 \quad \text{for } 0 \leq x \leq \pi,$$

$$\frac{\partial U}{\partial t}(x, 0) = 0 \quad \text{for } 0 \leq x \leq \pi.$$

Solution

We begin by letting

$$U_0(x, t) = \frac{R \sin \frac{\omega x}{c} \sin \omega t}{\sin \frac{\omega \pi}{c}},$$

which we know from Exercise 1 satisfies the wave equation and the boundary conditions but *not* the given initial conditions.

Then we put

$$V(x, t) = U(x, t) - U_0(x, t).$$

Putting $t = 0$, and using the notation of Subsection 3.3, we have

$$V(x, 0) = \phi(x)$$

where

$$\begin{aligned} \phi(x) &= U(x, 0) - U_0(x, 0) \\ &= 0 \quad \text{for } 0 \leq x \leq \pi. \end{aligned}$$

Also,

$$\frac{\partial V}{\partial t}(x, 0) = \psi(x)$$

where

$$\begin{aligned} \psi(x) &= \frac{\partial U}{\partial t}(x, 0) - \frac{\partial U_0}{\partial t}(x, 0) \\ &= 0 - \frac{R\omega \sin \frac{\omega x}{c}}{\sin \frac{\omega \pi}{c}}. \end{aligned}$$

Using the box on page 24, we have

$$V(x, t) = \sum_{n=1}^{\infty} \sin nx (A_n \cos nct + B_n \sin nct)$$

where

$$A_n = 0 \quad \text{for all } n \text{ (as } \phi \text{ is the zero function)}$$

and

$$\begin{aligned} B_n &= \frac{2}{n\pi c} \int_0^{\pi} \psi(x) \sin nx \, dx \\ &= \frac{-2R\omega}{n\pi c \sin \frac{\omega \pi}{c}} \int_0^{\pi} \sin \frac{\omega x}{c} \sin nx \, dx \\ &= \frac{-R\omega}{n\pi c \sin \frac{\omega \pi}{c}} \left[\frac{\sin \left(\frac{\omega}{c} - n \right) x}{\frac{\omega}{c} - n} - \frac{\sin \left(\frac{\omega}{c} + n \right) x}{\frac{\omega}{c} + n} \right]_0^{\pi} \end{aligned}$$

(using the formula for integrating $\sin \alpha x \sin \beta x$, on page 16 of the *Handbook*)

$$= \frac{-R\omega}{n\pi c \sin \frac{\omega \pi}{c}} \left[\frac{\left(\frac{\omega}{c} + n \right) \sin \left(\frac{\omega}{c} - n \right) x - \left(\frac{\omega}{c} - n \right) \sin \left(\frac{\omega}{c} + n \right) x}{\left(\frac{\omega}{c} + n \right) \left(\frac{\omega}{c} - n \right)} \right]_0^{\pi}$$

$$\begin{aligned}
&= \frac{-R\omega}{n\pi c \sin \frac{\omega\pi}{c}} \left[\frac{\frac{\omega}{c} \left(\sin \left(\frac{\omega}{c} - n \right) x - \sin \left(\frac{\omega}{c} + n \right) x \right) + n \left(\sin \left(\frac{\omega}{c} - n \right) x + \sin \left(\frac{\omega}{c} + n \right) x \right)}{\left(\frac{\omega}{c} \right)^2 - n^2} \right]_{\pi}^0 \\
&= \frac{-R\omega}{n\pi c \sin \frac{\omega\pi}{c}} \left[\frac{-2 \frac{\omega}{c} \cos \frac{\omega x}{c} \sin nx + 2n \sin \frac{\omega x}{c} \cos nx}{\left(\frac{\omega}{c} \right)^2 - n^2} \right]_{\pi}^0
\end{aligned}$$

(using the trigonometric formula for $\sin(\alpha + \beta)$ on page 8 of the *Handbook*, letting

$$\alpha = \frac{\omega}{c}, \beta = \pm n)$$

$$\begin{aligned}
&= \frac{-2R\omega c}{\pi(\omega^2 - n^2 c^2) \sin \frac{\omega\pi}{c}} \left(\sin \frac{\omega\pi}{c} \cos n\pi \right) \\
&= \frac{2R\omega c (-1)^{n+1}}{\pi(\omega^2 - n^2 c^2)}.
\end{aligned}$$

Finally we have for our solution for V ,

$$V(x, t) = \frac{2R\omega c}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\omega^2 - n^2 c^2} \sin nx \sin nct$$

and our solution for $U(x, t)$ is

$$U(x, t) = \frac{R \sin \frac{\omega x}{c} \sin \omega t}{\sin \frac{\omega\pi}{c}} + \frac{2R\omega c}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\omega^2 - n^2 c^2} \sin nx \sin nct.$$

This then is the solution of our problem (which is referred to in the television programme), but you should notice that it is certainly not valid if $\sin \frac{\omega\pi}{c} = 0$, in other words if ω is some multiple of c . In fact if ω is close to some multiple of c then the function

$$U_0(x, t) = \frac{R \sin \frac{\omega x}{c} \sin \omega t}{\sin \frac{\omega\pi}{c}}$$

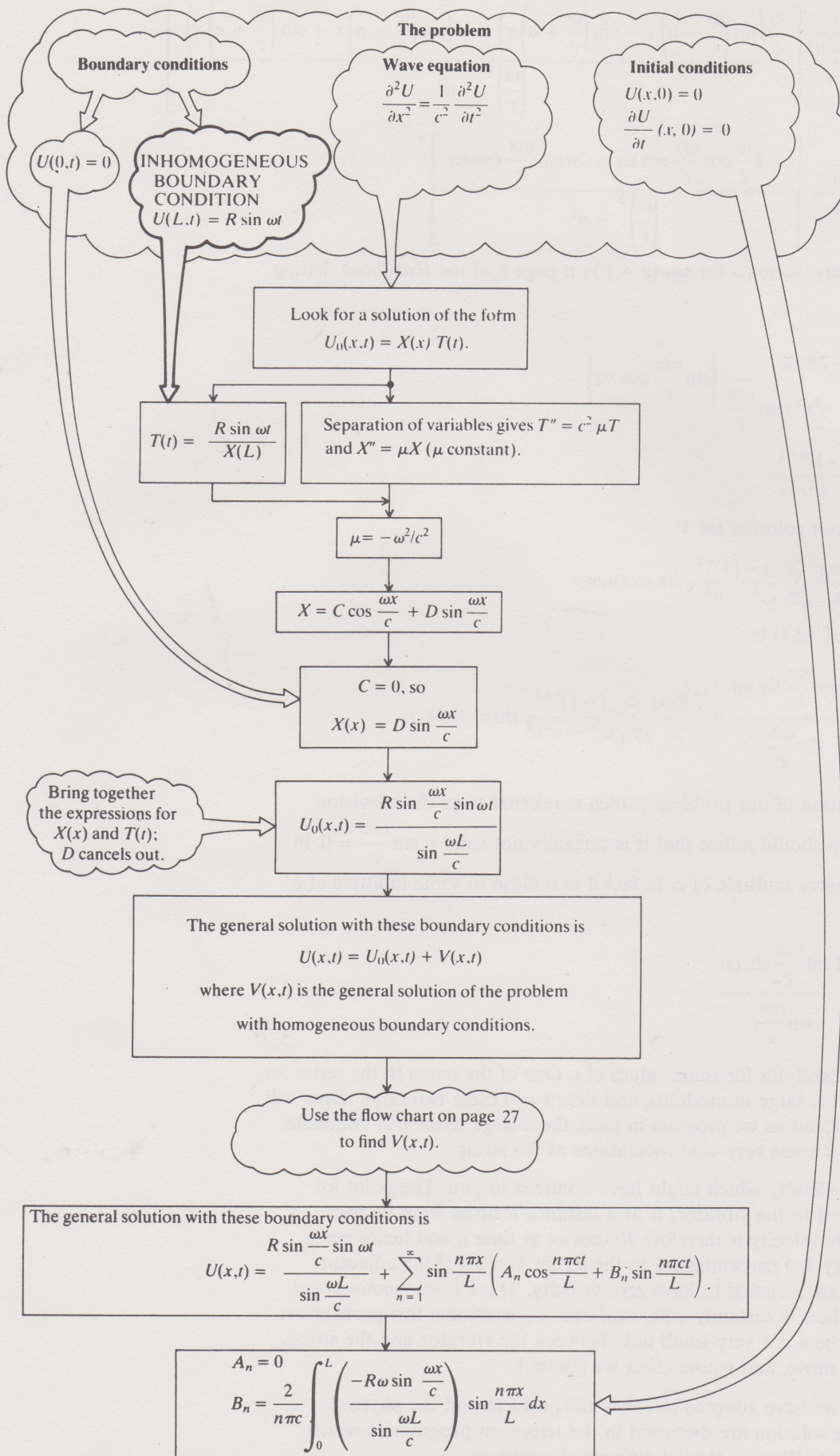
will be very large in modulus for some values of t . One of the terms in the series for $V(x, t)$ will also be very large in modulus, and near $t = 0$ these two large terms will effectively cancel out, but as we progress in time, these large terms will dominate, and we should expect some very wild oscillations of the string.

(There is a further difficulty which might have occurred to you. The point for which $x = \pi$ (attached to the vibrator) is at a distance $R \sin \omega t$ from its rest position at time t . Its velocity is therefore $R\omega \cos \omega t$ at time t , and hence when $t = 0$ it has a velocity $R\omega$ perpendicular to the string. But all of the adjacent points of the string are assumed to have zero velocity. There is no *mathematical* difficulty here, but there is certainly a *physical* one. To overcome this problem we might assume that there is a very small link, between the vibrator and the string, which is allowed to move, and whose effect we ignore.)

The strategy which we have adopted to solve this problem and the physical interpretation of the solution are discussed in the television programme which accompanies this unit, 'Waves—Partial differential equations'.

Summary of Section 4

The flow chart overleaf is intended to provide a summary of this section.



5 End of unit problems

In this set of problems you will be led through an investigation of the vibration of a uniform heavy spring suspended vertically from a fixed point with a weight of mass M attached to its lower end. Later for simplicity we assume that $M = 0$.

We use the notation of Example 1 of Section 2 (page 10), and suppose that the spring is of unstretched length l_0 , modulus k and mass M_0 , and that a mass M is attached to its lower end.

The element between P and Q is presumed to have moved to become an element of length δs at time t , and P is then at a distance s from A .

You may attempt this set of problems in the following form, or if you find it too difficult you may work through the sequence of questions given after the problems, which take you through the working in easy stages.

Problem 1

With the above notation and assumptions, show that

$$k \frac{\partial^2 s}{\partial x^2} + \frac{M_0 g}{l_0} = \frac{M_0}{l_0} \frac{\partial^2 s}{\partial t^2}.$$

Problem 2

Find a solution $s(x, t)$ of the above partial differential equation which is independent of time (that is $s(x, t) = X(x)$).

Problem 3

Assuming now that $M = 0$:

- show that the tension in the spring at B is zero;
- use Hooke's law to show that

$$\frac{\partial s}{\partial x}(l_0, t) = 1 \text{ for all } t;$$

- find the general solution of the above partial differential equation in the form of a series, subject to the boundary conditions

$$\left. \begin{aligned} s(0, t) &= 0 \\ \frac{\partial s}{\partial x}(l_0, t) &= 1 \end{aligned} \right\} \text{ for all } t.$$

(Hint: Consider the difference between the required solution and the solution which is independent of time.)

[Solutions to Problems 1–3 on pp. 36–8]

Sequence of questions

Our objective is to find a partial differential equation involving $s(x, t)$, the distance of P from A at time t , and we shall require the tension in the spring at P , which we denote by T . Read again through Example 1 of Subsection 2.1 (page 10) and then answer the following questions.

- What is the mass of the element δs ?
- What forces are acting on the element δs ?
- What is the total downward force acting on the element δs ?
- Write down expressions for:
 - the distance of P from A ,
 - the velocity of P downwards,
 - the acceleration of P downwards,
 at time t .
- Use the answer to Question (d) to write down an equation derived from Newton's second law

$$\text{force} = \text{mass} \times \text{acceleration}$$

involving T and $\frac{\partial^2 s}{\partial t^2}$. Take the limit of this equation as $\delta x \rightarrow 0$.

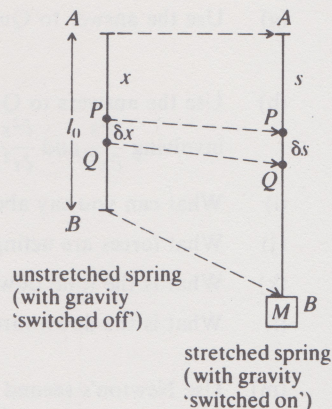


Figure 1. A spring in its unstretched and stretched states.

- (f) Use Hooke's law to write down an expression for T in terms of $\frac{\partial s}{\partial x}$. (See Subsection 2.1.)
- (g) Use the answer to Question (f) to find an expression for $\frac{\partial T}{\partial x}$ in terms of $\frac{\partial^2 s}{\partial x^2}$.
- (h) Use the answers to Questions (e) and (g) to write down a partial differential equation involving $\frac{\partial^2 s}{\partial x^2}$ and $\frac{\partial^2 s}{\partial t^2}$ (i.e. eliminate $\frac{\partial T}{\partial x}$).
- (i) What can you say about $s(0, t)$?
- (j) What forces are acting on the mass M ?
- (k) What is the total downward force acting on the mass M ?
- (l) What is the downward acceleration of the mass M (in terms of the function s)?
- (m) Use Newton's second law to find a relationship between $T(l_0)$ and $\frac{\partial^2 s}{\partial t^2}(l_0, t)$.
- (n) What could you say about $T(l_0)$ if s were a function of x only?
- (o) What could you say about $\frac{\partial s}{\partial x}$ when $x = l_0$, if s were a function of x only?
- (p) Put $s = X(x)$ in the partial differential equation obtained in Question (h), and hence integrate twice to find a solution which is independent of time.
- (q) Do you recognize the solution of Question (p)?
(Look again at Example 1 of Section 2, page 10.)
- (r) Make a substitution

$$y = s(x, t) - \left(\frac{(M + M_0)g}{k} + 1 \right)x + \frac{M_0 g x^2}{2kl_0}$$

in the partial differential equation obtained in Question (h). What equation do you obtain?

- (s) What can you say about $y(0, t)$?
- (t) Assume now that $M = 0$. (The case $M \neq 0$ is too difficult for us at this stage.) What can you say about $T(l_0)$, using the answer to Question (m)?
- (u) What can you say about $\frac{\partial s}{\partial x}(l_0, t)$, using the answers to Questions (f) and (t)?
- (v) What can you say about $\frac{\partial y}{\partial x}(l_0, t)$, using the formula for y in Question (r), and the answer to Question (u)?
- (w) Use the method of Subsection 3.1 (considering three cases exactly as before) to find a set of solutions of the wave equation obtained in Question (r) with boundary conditions

$$\left. \begin{aligned} y(0, t) &= 0 \\ \frac{\partial y}{\partial x}(l_0, t) &= 0 \end{aligned} \right\} \text{ for all } t,$$

these being the boundary conditions derived as answers to Questions (s) and (v).

Appendix: Solutions to the exercises

Solutions to the exercises in Section 1

1.(i) $F(x, y) = \frac{x}{y}$, so

$$\frac{\partial F}{\partial x} = \frac{1}{y}, \quad \frac{\partial F}{\partial y} = -\frac{x}{y^2}.$$

(ii) $F(x, y) = Ax^2 + 2Bxy + Cy^2$, so

$$\frac{\partial F}{\partial x} = 2Ax + 2By, \quad \frac{\partial F}{\partial y} = 2Bx + 2Cy.$$

(iii) $F(x, y) = \sin(2x + y)$, so

$$\frac{\partial F}{\partial x} = 2\cos(2x + y), \quad \frac{\partial F}{\partial y} = \cos(2x + y).$$

(iv) $F(x, y) = \phi(2x + y)$, so

$$\begin{aligned} \frac{\partial F}{\partial x} &= 2\phi'(2x + y) & (\text{where } \phi' \text{ denotes the} \\ & & \text{derived function of } \phi), \\ \frac{\partial F}{\partial y} &= \phi'(2x + y). \end{aligned}$$

(v) $F(x, y) = \phi(ax^2 + by^2)$, so

$$\frac{\partial F}{\partial x} = 2ax\phi'(ax^2 + by^2), \quad \frac{\partial F}{\partial y} = 2by\phi'(ax^2 + by^2).$$

2.(i) $f'(x)$

(ii) $\frac{\partial F}{\partial x}(x, t)$

(iii) $\frac{\partial^2 F}{\partial x^2}(x, t)$

3.(i) $F(x, y) = xy$, so $\frac{\partial F}{\partial x} = y$, $\frac{\partial F}{\partial y} = x$, and thus

$$\frac{\partial^2 F}{\partial x^2} = 0, \quad \frac{\partial^2 F}{\partial y \partial x} = 1,$$

$$\frac{\partial^2 F}{\partial x \partial y} = 1, \quad \frac{\partial^2 F}{\partial y^2} = 0.$$

(ii) $F(x, y) = x^3 + 2x^2y + y^3$, so

$$\frac{\partial F}{\partial x} = 3x^2 + 4xy, \quad \frac{\partial F}{\partial y} = 2x^2 + 3y^2, \text{ and thus}$$

$$\frac{\partial^2 F}{\partial x^2} = 6x + 4y, \quad \frac{\partial^2 F}{\partial y \partial x} = 4x,$$

$$\frac{\partial^2 F}{\partial x \partial y} = 4x, \quad \frac{\partial^2 F}{\partial y^2} = 6y.$$

(iii) $F(x, y) = e^{x+y}$, so

$$\frac{\partial F}{\partial x} = e^{x+y}, \quad \frac{\partial F}{\partial y} = e^{x+y}, \text{ and thus}$$

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y^2} = e^{x+y}.$$

Indeed, all the partial derivatives of F , of all orders, are equal to e^{x+y} .

4.(i) $U = f(x)g(y)$, so

$$\frac{\partial U}{\partial x} = f'(x)g(y), \quad \frac{\partial U}{\partial y} = f(x)g'(y), \quad \frac{\partial^2 U}{\partial x \partial y} = f'(x)g'(y).$$

Thus, $\frac{\partial U}{\partial x} \frac{\partial U}{\partial y} = f(x)g(y)f'(x)g'(y)$

$$= U \frac{\partial^2 U}{\partial x \partial y}.$$

(ii) $U = f(ay - bx)$, so

$$\frac{\partial U}{\partial x} = -bf'(ay - bx) \quad \text{and} \quad \frac{\partial U}{\partial y} = af'(ay - bx).$$

Thus, $a \frac{\partial U}{\partial x} + b \frac{\partial U}{\partial y} = 0$.

(iii) $y = f(x - t) + g(x + t)$, so

$$\frac{\partial y}{\partial x} = f'(x - t) + g'(x + t),$$

$$\frac{\partial y}{\partial t} = -f'(x - t) + g'(x + t), \quad \text{giving}$$

$$\frac{\partial^2 y}{\partial x^2} = f''(x - t) + g''(x + t) \quad \text{and}$$

$$\frac{\partial^2 y}{\partial t^2} = f''(x - t) + g''(x + t).$$

Thus, $\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$.

5. (i) and (iv) are homogeneous linear. (ii) fails to be linear because of the term $x \frac{\partial F}{\partial x} \frac{\partial F}{\partial y}$, in which two partial derivatives are multiplied together, while (iii) fails to be homogeneous because of the term e^{xy} , which does not involve the unknown or any of its derivatives. The only one of the four which is constant-coefficient is (iv).

Solution to the exercise in Section 2

1. Substituting $s = U + x$ into Equation (13) gives

$$\frac{\partial^2}{\partial x^2}(U + x) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2}(U + x).$$

Thus

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 x}{\partial x^2} = \frac{1}{c^2} \left(\frac{\partial^2 U}{\partial t^2} + \frac{\partial^2 x}{\partial t^2} \right).$$

Now, $\frac{\partial x}{\partial x} = 1$, so $\frac{\partial^2 x}{\partial x^2} = 0$. Also $\frac{\partial x}{\partial t} = 0$ since $\frac{\partial}{\partial t}$ means the rate of change as t varies and x remains fixed. Thus $\frac{\partial^2 x}{\partial t^2} = 0$, and we obtain Equation (14).

Solutions to the exercises in Section 3

1.(i) Substituting $c = 2$, $\mu = 1$ gives

$$T''(t) = 4T(t) \text{ and } X''(x) = X(x).$$

From Unit 6, Section 1, we have

$$T = Ae^{2t} + Be^{-2t} \text{ for some constants } A \text{ and } B;$$

also

$$X = Ce^x + De^{-x} \text{ for some constants } C \text{ and } D.$$

(ii) Substituting $c = 2$, $\mu = -1$ gives

$$T'' = -4T \quad \text{and} \quad X'' = -X.$$

Hence

$$T = A \cos 2t + B \sin 2t$$

$$\text{and } X = C \cos x + D \sin x$$

for some constants A , B , C and D .

(iii) Substituting $c = -2$, $\mu = 1$ gives

$$T'' = 4T \quad \text{and} \quad X'' = X,$$

the same equations as in part (i). Thus the solution is the same as in part (i). This shows that although the sign of μ is important, the sign of c is irrelevant.

2. We know that $Ae^{ht} + Be^{-ht} = 0$ for all values of t , so that, in particular, it is true for $t = 0$ and $t = 1/h$. Therefore

$$A + B = 0$$

$$Ae + \frac{B}{e} = 0$$

and this pair of equations has the unique solution

$$A = B = 0.$$

3. If

$$\theta(\dot{x}, t) = X(x)T(t),$$

then differentiating partially with respect to x (for fixed t), we obtain

$$\frac{\partial^2 \theta}{\partial x^2} = X''(x)T(t)$$

and differentiating partially with respect to t (for fixed x),

$$\frac{\partial \theta}{\partial t} = X(x)T'(t).$$

It follows that

$$X''(x)T(t) = c^2 X(x)T'(t),$$

so that

$$\frac{X''(x)}{c^2 X(x)} = \frac{T'(t)}{T(t)} = \mu, \text{ say.}$$

Using the same argument as before we deduce that μ is constant, and hence the functions X and T satisfy

$$X'' = c^2 \mu X \quad \text{and} \quad T' = \mu T.$$

4. Using the formula for A_n , we have

$$A_n = \frac{2}{\pi} \int_0^\pi \phi(x) \sin nx \, dx.$$

Using the hint (with l set equal to π), we obtain

$$A_n = \frac{8d}{n^2 \pi^2} \sin \frac{n\pi}{2}.$$

Also $B_n = 0$ for all n , since $\psi(x) = 0$, and it follows that

$$\begin{aligned} U(x, t) &= \frac{8d}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{2}}{n^2} \sin nx \cos nct \\ &= \frac{8d}{\pi^2} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m-1)^2} \sin (2m-1)x \cos (2m-1)ct. \end{aligned}$$

(This is the series used to model the motion of a vibrating string in the television programme 'Waves — Partial differential equations' which accompanies this text.)

Solution to the exercise in Section 4

1.(i) Using the boundary condition

$$U(L, t) = R \sin \omega t,$$

we have

$$X(L)T(t) = R \sin \omega t, \text{ so that}$$

$$T(t) = \frac{R \sin \omega t}{X(L)}. \quad (1)$$

(ii) Substituting the above expression for $T(t)$ into the equation $T''(t) = c^2 \mu T$, we have

$$-\frac{R\omega^2 \sin \omega t}{X(L)} = \frac{c^2 \mu R \sin \omega t}{X(L)}$$

$$\text{so that } \mu = -\frac{\omega^2}{c^2}.$$

(iii) Putting $\mu = -\frac{\omega^2}{c^2}$ in the equation $X'' = \mu X$, we have

$$X'' = -\frac{\omega^2}{c^2} X,$$

so that

$$X(x) = C \cos \frac{\omega x}{c} + D \sin \frac{\omega x}{c}. \quad (2)$$

(iv) If $U(0, t) = 0$ for all t , then $X(0) = 0$, and using Equation (2) we deduce that $C = 0$. Thus we have

$$X(x) = D \sin \frac{\omega x}{c}. \quad (3)$$

(v) $U(x, t) = X(x)T(t)$

$$= D \sin \frac{\omega x}{c} \left(\frac{R \sin \omega t}{D \sin \frac{\omega L}{c}} \right) \quad (\text{using Equations (1) and (3)})$$

$$= \frac{R \sin \frac{\omega x}{c} \sin \omega t}{\sin \frac{\omega L}{c}}.$$

Solutions to the problems and questions in Section 5

The solutions to the problems are set out below as answers to the sequence of questions. The answers to Questions (a) to (h) constitute the solution to Problem 1; from (i) to (p) constitutes the solution to Problem 2; and from (q) to (w) forms the solution to Problem 3.

1.(a) The element δs was originally of length δx , so that its mass is $\frac{M_0 \delta x}{l_0}$.

(b)(i) The tension upwards at P ;

(ii) the tension downwards at Q ;

(iii) the weight of the element δs acting downwards.

(c) $T(x + \delta x) - T(x) + \frac{M_0 g \delta x}{l_0}$.

(d)(i) $s(x, t)$.

(ii) $\frac{\partial s}{\partial t}(x, t)$ (abbreviated to $\frac{\partial s}{\partial t}$).

(iii) $\frac{\partial^2 s}{\partial t^2}(x, t)$ (abbreviated to $\frac{\partial^2 s}{\partial t^2}$).

(e) $T(x + \delta x) - T(x) + \frac{M_0 g \delta x}{l_0} = \frac{M_0 \delta x}{l_0} \frac{\partial^2 s}{\partial t^2}$,

and in the limit, after dividing by δx ,

$$\frac{\partial T}{\partial x} + \frac{M_0 g}{l_0} = \frac{M_0}{l_0} \frac{\partial^2 s}{\partial t^2}.$$

(f) $T = k \left(\frac{\partial s}{\partial x} - 1 \right)$.

(Use Equation (3) on page 10 with $T_0 = 0$, since the spring is originally unstretched; and use a partial derivative because s is a function of x and t .)

(g) $\frac{\partial T}{\partial x} = k \frac{\partial^2 s}{\partial x^2}$.

(h) $k \frac{\partial^2 s}{\partial x^2} + \frac{M_0 g}{l_0} = \frac{M_0}{l_0} \frac{\partial^2 s}{\partial t^2}$.

This is the solution to Problem 1.

2.(i) The top of the spring is fixed, so that $s(0, t) = 0$ for all t .

(j) Its weight Mg downwards and the tension in the spring upwards.

(k) $Mg - T(l_0)$.

(l) $\frac{\partial^2 s}{\partial t^2}(l_0, t)$, in other words $\frac{\partial^2 s}{\partial t^2}$ evaluated at (l_0, t) .

(m) $Mg - T(l_0) = M \frac{\partial^2 s}{\partial t^2}(l_0, t)$.

(n) $T(l_0) = Mg$, since in this case $\frac{\partial^2 s}{\partial t^2}$ would be zero.

(o) $\frac{\partial s}{\partial x}$ can be replaced by $\frac{ds}{dx}$ if s is a function of x only, and from the answers to Questions (f) and (n) we have

$$Mg = k \left(\frac{ds}{dx} - 1 \right)$$

when $x = l_0$.

(p) The equation of Question (h) becomes

$$k \frac{d^2 X}{dx^2} = -\frac{M_0 g}{l_0}$$

so that

$$k \frac{dX}{dx} = -\frac{M_0 g x}{l_0} + A.$$

Putting $x = l_0$ and using the answer to Question (o) we have

$$Mg + k = -M_0 g + A,$$

so that

$$A = (M + M_0)g + k$$

and therefore

$$k \frac{dX}{dx} = (M + M_0)g + k - \frac{M_0 g x}{l_0}.$$

Integrating again,

$$kX = ((M + M_0)g + k)x - \frac{M_0 g x^2}{2l_0} + B,$$

and from the answer to Question (i) we have $B = 0$, so that

$$X = \left(\frac{(M + M_0)g}{k} + 1 \right) x - \frac{M_0 g x^2}{2kl_0}.$$

This is the solution to Problem 2.

3.(q) This is precisely the solution which we obtained in Example 1 of Section 2 (see Equation (8) on page 12). In this context it is just what we should expect—namely the solution which is independent of time.

(r) The wave equation,

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2},$$

where $c^2 = \frac{kl_0}{M_0}$.

This is the hint suggested at the end of Problem 3.

(s) $y(0, t) = s(0, t) = 0$ for all t .

(t) $T(l_0) = 0$ because $M = 0$.

This is the solution to Problem 3(i).

(u) From the answers to Questions (f) and (t),

$$\frac{\partial s}{\partial x}(l_0, t) = 1 \text{ for all } t.$$

This is the solution to Problem 3(ii).

(v) Using the formula in Question (r) (and putting $M = 0$),

$$\frac{\partial y}{\partial x} = \frac{\partial s}{\partial x} - \left(\frac{M_0 g}{k} + 1 \right) + \frac{M_0 g x}{kl_0},$$

so that using the answer to Question (u),

$$\begin{aligned} \frac{\partial y}{\partial x}(l_0, t) &= 1 - \left(\frac{M_0 g}{k} + 1 \right) + \frac{M_0 g}{k} \\ &= 0 \quad \text{for all } t. \end{aligned}$$

(w) The discussion of Cases I and II is almost identical to that of Subsection 3.1, and there are no non-trivial solutions of either form. For Case III we have

$$y(x, t) = (C \cos hx + D \sin hx)T(t).$$

Since $y(0, t) = 0$ for all t , we must have $C = 0$ for a non-trivial solution.

Since $\frac{\partial y}{\partial x}(l_0, t) = 0$ for all t , we have

$$Dh \cos hl_0 = 0,$$

so that

$$hl_0 = \frac{\pi}{2} + n\pi$$

for some non-negative integer n .

The required set of solutions is therefore

$$y_n(x, t) = \sin(n + \frac{1}{2}) \frac{\pi x}{l_0} \left(A_n \cos(n + \frac{1}{2}) \frac{\pi ct}{l_0} + B_n \sin(n + \frac{1}{2}) \frac{\pi ct}{l_0} \right).$$

This is the solution to Problem 3(iii).

The general solution for $y(x, t)$ is therefore

$$\sum_{n=0}^{\infty} \sin(n + \frac{1}{2}) \frac{\pi x}{l_0} \left(A_n \cos(n + \frac{1}{2}) \frac{\pi ct}{l_0} + B_n \sin(n + \frac{1}{2}) \frac{\pi ct}{l_0} \right)$$

and so, undoing the substitution of Question (r) while remembering that now $M = 0$, the general solution for the original function $s(x, t)$ is

$$s(x, t) = \left(\frac{M_0 g}{k} + 1 \right) x - \frac{M_0 g x^2}{2kl_0} + \sum_{n=0}^{\infty} \sin(n + \frac{1}{2}) \frac{\pi x}{l_0} \left(A_n \cos(n + \frac{1}{2}) \frac{\pi ct}{l_0} + B_n \sin(n + \frac{1}{2}) \frac{\pi ct}{l_0} \right).$$

As in Subsection 3.3, a knowledge of the initial conditions would allow us to calculate the values of the A_n s and B_n s, thus solving as a Fourier series the problem of a heavy spring vibrating vertically under the effect of gravity.

